Period control of chaotic systems by optimization

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We introduce and implement a method of control based on optimization. An error function, which is a measure of deviation from the desired time evolution (periodic motion), is constructed. The error function is then minimized along the trajectory in order to stabilize an unstable periodic orbit. The process of optimization is not arbitrary but constrained to the dynamics. No specific knowledge of the desired state of the system is required. We also demonstrate how orbits of ''high'' period are similarly controlled. [S1063-651X(97)02001-1]

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I. INTRODUCTION

The term ''chaos'' is often used to describe low-dimensional, unpredictable time evolution accompanied by extreme sensitivity to initial data — the behavior common to many nonlinear dynamical systems. In many applications, regions of the parameter space where nonlinear effects are present are avoided, or the chaotic motion is eradicated by some large modification of the underlying system. Evidently, major modifications are costly and truncation of the parameter space may be too restrictive an approach. An alternative is to take advantage of the basic properties inherent in a chaotic system. A chaotic attractor, unlike a linear system in which a given parameter renders only one type of motion, possesses infinitely many periodic orbits, and many different time evolutions are simultaneously possible. Furthermore, the motion on the chaotic attractor is exponentially sensitive to small perturbations. Ott, Grebogi, and Yorke (OGY) [1] illustrated not only that chaotic systems may be controlled, but that the richness of possible behaviors in chaotic systems may be exploited to enhance the performance of a dynamical system in a manner that would not be possible had the system's evolution not been chaotic. Shortly after this publication, Ditto *et al.* [2] reported a successful laboratory implementation of the control strategy outlined in [1], demonstrating that controlling chaos is not just in theory, but physically attainable as well. The method of OGY [1] and its variations [3] control a chaotic dynamical system by first identifying a periodic orbit and applying small perturbations to a system parameter p to stabilize the unstable periodic orbit. It was later shown [4,5] by methods unrelated to OGY that the system may also be controlled by perturbing the state variables directly. However, while the latter approach does not result in a goal oriented control and the final state is periodic but otherwise arbitrary, the former results in a desired periodic orbit and we will refer to it in the following sections.

The following question motivates our method: is a goal oriented control without any explicit knowledge of the goal orbit possible? We believe the answer is yes and the goal oriented control may be accomplished by controlling the period and not the state. The relevance of this question is evident since in many engineering applications the goal is to convert chaotic time evolution to a periodic motion of a specific period, i.e., period control. Furthermore, identifying a periodic orbit, either by observation or by computations, may not be practical (high dimensional systems) or the periodic orbits may not be fixed objects of the phase space (slowly driven systems). There are also systems similar to the heart [6], where the phase space position as well as the type of the periodic orbits may change depending on the operational requirements. Often, in the case of biological and chemical systems, a control parameter p is not accessible. Control of all such systems, we believe, may be accomplished within the framework of optimization, as illustrated below.

II. THE METHOD

We perturb the system slightly to accomplish the control task and the small perturbations we require (unlike the method of [5]) are specific and based on a goal oriented strategy. To simply illustrate the point, we chose a map-based time evolution $x_{n+1}=F(x_n,p)$. Our strategy, rooted in optimization, consists in minimizing a positive definite *error* function given by

$$g_n=g(\mathbf{x}_n,\mathbf{p}_n), \tag{1}$$

where $\mathbf{x}_n$ are the state variables and $\mathbf{p}_n$ the system parameters which may or may not be time dependent. For simplicity, consider a scenario where a steady state (x^*,p^*) behavior of an otherwise chaotic system is desired. For now we ignore the parameter p (see Sec. IV). The *error* function is constructed by embodying a general attribute of the desired orbit into an expression, the evolution of which is constrained by the dynamics. Here, a general attribute of the desired orbit (a fixed point in this case) is that $F[x_n]=x_n$ for all $n\in\mathbf{N}$. Consider then, as a measure of the *error,*

$$g(x_n)=[F(x_n)-x_n]^2, \tag{2}$$

which is clearly minimized when the system reaches a steady state. Many other measures of error can be defined, such as $[f(x_n)-x^*]^2$ or $(x_n-x^*)^2$, and in general a given measure of the error will give a different controller. But for the purposes of this paper we will assume that we do not want to use (or do not know) any specific information about the periodic orbits in order to control them, therefore, we will concentrate on the error measure given by Eq. (2). To formulate the problem in the context of optimal control, we need to find a sequence of small perturbations, $\boldsymbol{\epsilon}_n$, so that a suitable performance index is minimized. In the presence of small perturbations the dynamical system is then given by

$$x_{n+1}=F(x_n)+\epsilon_n. \tag{3}$$

The magnitude $g(x_n)$ is used as a flag for when the control is to be applied [$g(x_n)$ is small] and when it should be relinquished [$g(x_n)$ is large]. This defines the threshold condition $|g(x_n)|<\Gamma$, where Γ is chosen so that the control is kept ''small,'' a premise implicit in small perturbations control. The form of our *error* dictates a quadratic optimal control approach, whereby a quadratic performance index,

$$J=\frac{1}{2}\sum_{n=0}^{N\to\infty}[g(x_n)+\epsilon_n B\epsilon_n]=\sum_{n=0}^{N\to\infty} j_n, \tag{4}$$

is minimized to determine the sequence ϵ_n. In general, B is a matrix specifying how the various degrees of freedom are combined in the construction of a performance index.

Before resorting to the formal methods of optimal control theory (presented in the Sec. IV), we minimize Eq. (4) without any constraints. Considering this question is justified since $g(x_n)$ may be thought of as a ''quasipotential'' [7], the minimization of which yields the sequence of steps, the path in the phase space, needed to go from x_n to x^*. Minimizing Eq. (4) free of constraints leads to the following set of equations

$$\frac{\partial j_n}{\partial x_n}\delta x_n+\frac{\partial j_n}{\partial \epsilon_n}\delta\epsilon_n=0. \tag{5}$$

Intuitively, we know near the fixed point the perturbations are proportional to the state variable, $\delta\epsilon_n \propto \delta x_n$ (in the limit of small perturbations). This assumption leads one to the conclusion that

$$\epsilon_n \approx \alpha\nabla g(x_n). \tag{6}$$

Combining Eqs. (3) and (6) shows that, given the state variable x_n that admits a small $g(x_n)$, the dynamical system can be directed to its fixed point by

$$x_{n+1}=F(x_n)+\alpha\nabla g(x_n), \tag{7}$$

where α can be a matrix if required. To lend credence to Eq. (7), let us examine the resulting perturbed system near the fixed point, $x_n \approx x^*$. It is important to note that we do not require any specific knowledge of the fixed point, therefore, we expand the perturbed system of Eq. (7) about the point x_{n-1} and denote the deviation by δx_n. We obtain $\delta x_{n+1}=[DF(x_n)+\alpha D(\nabla g)(x_n)]\delta x_n$. The control is achieved by $\delta x_{n+1}/\delta x_n \to 0$ with increasing n which constrains α (a matrix in general) such that $|DF(x_n)+\alpha D(\nabla g)(x_n)|<1$. A value of α can then be obtained from this constraint equation, since it is linear in α. Furthermore, the correction (feedback error) very close to the fixed point is actually proportional to the deviation δx_n, as (intuitively) expected, but turns into a nonlinear perturbation away from the fixed point.

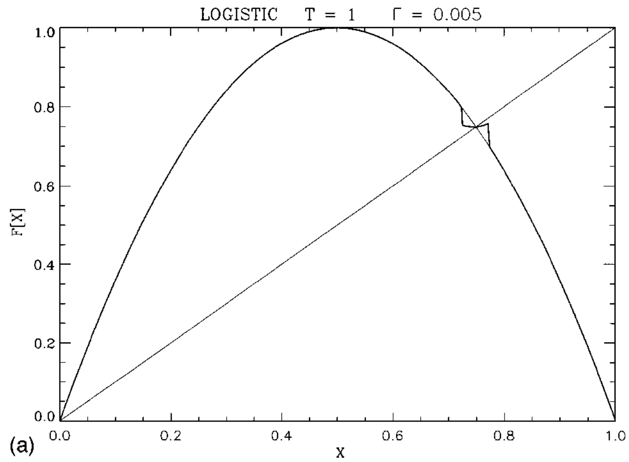
It is very important to realize that depending on the performance index we choose, we obtain different controllers, e.g., $\epsilon_n=\alpha(x_n-x^*)$, $\epsilon_n=\alpha[F(x_n)-x^*]$, and so on. In 1992, Pyragas [4] suggested an interesting feedback mechanism, whereby the past history of the system was used to stabilize the system to its fixed point, $x_{n+1}=M(x_n)+k(x_{n-1}-x_n)$. There are two relevant issues of this publication that we must discuss. First, the final form of the algorithm mentioned by the author ([4]) can be obtained by optimizing a proper performance index. Second, after the appearance of [4] a number of investigators successfully applied the control method (described therein) to real circuits modeled by ordinary differential equations, which suggests that the algorithm given by Eq. (7) may also be effective in ''real'' applications. However, there are differences between the method developed in this paper and that of [4] besides the different approaches in obtaining the respective controllers. The specific algorithm of Eq. (7) does not increase the dimension of the system, keeping the tractability and hence the analysis of the problem manageable, whereas the algorithm suggested in [4] increases the dimension of the original system. Equation (7) may naturally be generalized to stabilize the systemq to any desired period (see below), however by the form of the control suggested [$\epsilon_n=k(x_{n-1}-x_n)$ [4]], an $(m+1)$-dimensional system has to be considered when stabilizing a period-m orbit.

III. ILLUSTRATIONS AND EXAMPLES (PROOF OF PRINCIPLE)

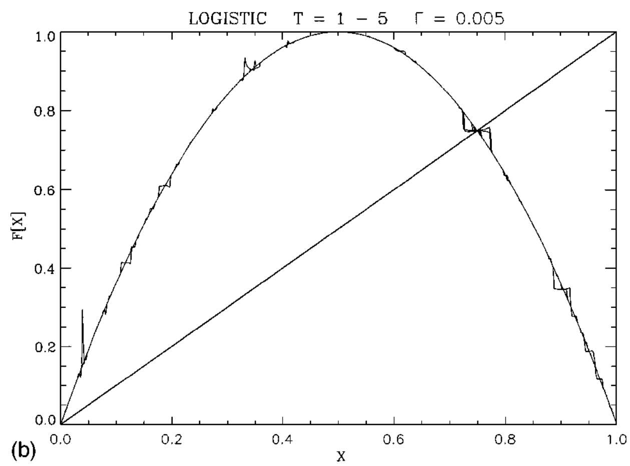
As a simple yet illustrative example, we apply our method to the logistic map $F(x_n,\nu)=\nu x_n(1-x_n)$, for $\boldsymbol{\nu}=4$ the map is chaotic. This map is a simple model for the time evolution of a certain insect population (May [16]). The period-1 orbit (fixed point) of this map is $x^*=\frac{3}{4}$ and $|F'(x^*,\nu)|>1$, hence x^* is unstable and repels the nearby points. When $g(x_n)\leqslant\Gamma$, the threshold, the control algorithm perturbs $F(x_n)$ to $H(x_n)=F(x_n)+\epsilon_n$ (see Sec. IV). Figure 2(a) shows the stabilized period-1 orbit. In accordance with small perturbations control, Γ is kept small and α is chosen with prudence to constrain the size of ϵ_n. The effect of feedback is best illustrated by superposing $F(x_n)$ and $H(x_n)$, as shown in Fig. 1(a), and the perturbation is shown to be small. To examine the role α plays, consider the expression for stability

$$|DF(x^*)+\alpha D(\nabla g)(x^*)|<1, \tag{8}$$

which can be solved to determine the required strength of the coupling, hence for all $\alpha\in(\frac{1}{18},\frac{3}{18})$ the system [Eq. (7)] converges to the fixed point of the unperturbed map. The advantage of our method is most significant when employed to stabilize high period orbits. Imagine an application for which switching between orbits of different periods is required. Of the many periodic orbits embedded in an attractor, there may exist many with the same period, each having its own basin of attraction. Here we do not discriminate between the different periodic orbits of the same period; we only require the period to be specific. Therefore, when the control is applied, $\alpha\neq0$, and a period-m orbit stabilized, it may be any one of the possible orbits of period m, depending on the basin of attraction that was visited first. The nonspecific nature of the orbit enlarges the set of suitable α values. Specifically, let



(a)



(b)

FIG. 1. (a) The logistic map, perturbed so as to stabilize the period $m=1$ orbit. α is set to a nonzero value when $g(x_n)\leqslant\Gamma=0.005$. (b) The superposition of the maps $H(x)$ required to control periods 1 to 5 for the logistic map. The small features are the deviations from the unperturbed map and are caused by the perturbations of the control signal.

I_α^m be the interval containing all values of α for which a period m orbit is stabilized. Then,

$$I_\alpha^m=\bigcup_{i=1}^{M} I_{\alpha_i}^m, \tag{9}$$

where $I_{\alpha_i}^m$ is the interval of α that stabilizes the ith period m orbit. To illustrate how a high period orbit may be controlled, consider the case where an unstable period m orbit of a one-dimensional map is to be stabilized. A suitable *error* functional, assuming again no explicit knowledge of the periodic orbits, is

$$g(x_n)=|F^m(x_n)-x_n|^2. \tag{10}$$

Control is achieved by allowing the dynamics to minimize the *error*, as was done for the period-1 orbit. Evidently, the algorithm does not require specific information about the desired orbit. The dynamics finds a suitable periodic orbit and no tracking of the high period orbit is necessary, hence the information overhead is minimal. We choose the logistic map as a ''proof of principle'' demonstration and control it to execute a series of switching by the successive stabilization of periodic orbits of period 1 to period 5 [Fig. 2(b)]. The only thing needed to carry the switching between orbits of different periods is the number m. When the dynamical equations are not known or when computing $I_{\alpha_j}^i$ is too cumbersome, $I_{\alpha_j}^i$ may be found by trial and error. However, it may be more efficient to estimate the equations governing the system dynamics near the desired periodic orbits [see remark (3)], and approximate a suitable α. Figure 1(b) illustrates a superposition of the perturbed maps for all periods, from $m=1$ to $m=5$. It is clear that the actual map is not significantly altered by the application of these perturbations. The required control strength is shown in Fig. 2(c).

IV. DISCUSSION

A. A brief discussion on parametric control

OGY type of control may be considered as a special case of optimization. Parameter p can be treated as a system variable p_n [8]. By using appropriate performance indexes we can obtain more traditional controllers such as $p_{n+1}=p^*-\alpha(x_n-x^*)$, $p_{n+1}=p^*-\alpha[F(x_n)-x_n]$, and so on, where p^* is the goal parameter. For illustration, we took an appropriate performance index to obtain $p_{n+1}=p^*-\alpha[dg(x_n,p_n)/dp_n]$ and applied it to stabilize the Henon map (and others that will be presented elsewhere [9]),

$$x_{n+1}=1.3-x_n^2+p_n^* y_n, \tag{11}$$

$$y_{n+1}=x_n. \tag{12}$$

For $p^*=0.3$, as shown in Fig. 3, the x component of the Hennon map [Fig. 3(a)] along with the control strength [Fig. 3(b)] are plotted. The parameter α is obtained from the constrained equation, a 3×3 matrix, which is linear in α making its solution tractable. The extension to higher dimensions is again trivial, though a little more involved.

B. Minimization constrained by a rule

Let us now consider the formal approach to minimization that is constrained by a rule. The minimization of Eq. (4) subject to a constraint, Eq. (3), can be solved by incorporating the constraint in the function to be minimized by the use of Lagrange multipliers [10]. Using the Lagrange multipliers, $\lambda(1)$, $\lambda(2)$, . . . we define a new performance index

$$P=\sum_{n=0}^{N}\left[g(x_n)+\frac{\epsilon_n B\epsilon_n}{2}+\lambda(n+1)[F(x_n)+\epsilon_n-x_{n+1}]\right], \tag{13}$$

the minimization of which requires the solution of the following expressions:

$$\frac{\partial P}{\partial x_i}=\nabla g(x_n)+\lambda_{i+1}DF(x_i)-\lambda_i=0, \tag{14}$$

$$\frac{\partial P}{\partial \epsilon_i}=B\epsilon_i+\lambda_{i+1}=0, \tag{15}$$

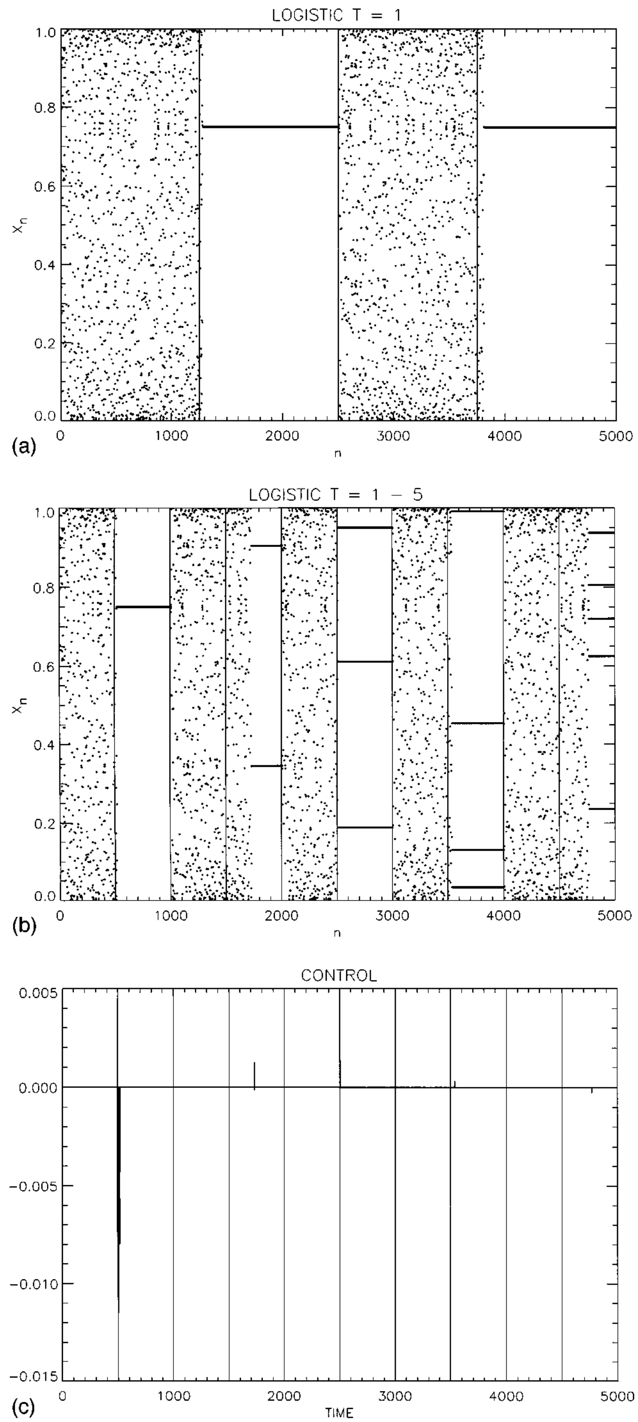


FIG. 2. (a) The period $m=1$ orbit of the logistic map is stabilized. The thin vertical lines (running top to bottom) indicate the iterate number at which the control was turned on, $\alpha\neq 0$ and off, $\alpha=0$. (b) Period 1 to period 5 (left to right) of the logistic map are controlled. (c) The time history of the control signal, always smaller than 2% of the size of the attractor.

$$\frac{\partial P}{\partial \lambda_i}=F(x_i)+\epsilon_i-x_{i+1}=0, \tag{16}$$

where $DF[x]$ is the Jacobian. This system may be solved to determine the sequences of corrections (perturbations) that in the limit of $N\rightarrow\infty$ leads to the desired periodic motion.

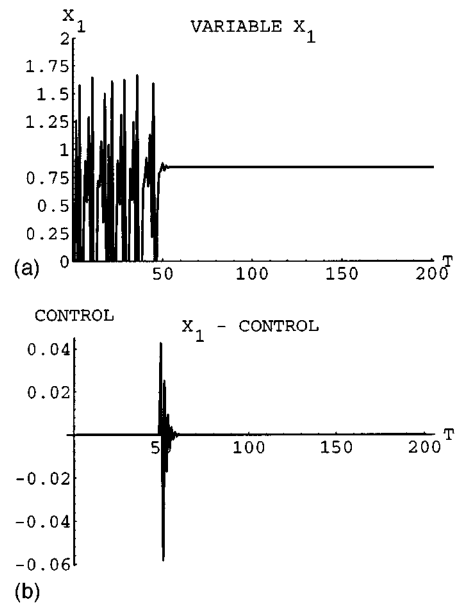


FIG. 3. (a) The x component of the controlled Henon map as a function of time. (b) The size of the control.

Equation (16) simply expresses the time evolution in the presence of the correction, Eq. (15) relates the Lagrange multipliers to the corrections, and Eq. (14) determines the sought after sequence, and in fact deserves a closer look since it contains information about a new feature of the dynamics, not shared by the unperturbed map. Solving Eq. (15) and substituting the result in Eq. (14) indicates how one may go about calculating the perturbation sequence. Traditionally, in applying the optimal control method to a linear problem, N is determined in advance and a variation with respect to the final state, $x(N)$, determines the final value of the Lagrange multipliers, λ_N. Knowing λ_N, Eqs. (14) and (15) are then used to compute the correction sequence in advance and apply the results when the system is in operation, which leads to the desired final state [11]. In the present application, however, we consider the steady state form of the optimal control, $N\rightarrow\infty$. To eliminate the Lagrange multipliers from the expressions, Eqs. (14) and (15) are combined to give

$$\epsilon_i=\frac{1}{B}\nabla g(x_n)\frac{1}{DF[x_i]}+\epsilon_{i-1}\frac{1}{DF[x_i]}, \tag{17}$$

giving a 2D system where x_{i+1} is given by Eq. (16). To examine the stability of the above system, the eigenvalues of the Jacobian for Eq. (8) evaluated at the final state are needed. The performance index [Eq. (2)] leads to the fixed point of the map, x^*. For the particular case of a 1D map, the Jacobian of this 2D system is

$$\mathbf{J}=\begin{pmatrix} d+f\dfrac{e^2}{d} & 1+\dfrac{1}{f}\left(\dfrac{d}{e}\right)^2 \\ \dfrac{f}{d^2}[(ed)^2+fe^4+e^2] & \dfrac{f}{d}\left[e^2+\dfrac{1}{f}\left(\dfrac{d}{e}\right)^2+\dfrac{1}{f}\right] \end{pmatrix}, \tag{18}$$

where $d = F'(x') \neq 0$, $e = F(x') - 1$, and $f = 1/B$. For a simple map, such as the logistic map, we can solve the characteristic equation $\det(J - 1\nu) = 0$ to determine the eigenvalues. This yields intervals (of nonzero measure) of f on the real line where both $|\nu_1|$ and $|\nu_2|$ are less than 1 and hence convergence to the fixed point [9]. Even though the above method is stable and leads to the desired final state, there is yet a simplification that leads to a less cumbersome prescription for implementing the control algorithm. Loosely speaking, once we have $|\epsilon_i|$ small, we want to force $|\epsilon_{i+1}| \to 0$, which modifies the control algorithm to be

$$\epsilon_i \approx \frac{1}{B} \nabla g(x_n), \tag{19}$$

which is the same result as before.

For the case when the system's equations are not known *a priori*, and reconstructing a map from data is eminent, monitoring $g(x_n)$ as opposed to its derivative might become an issue, and then an appropriate error measure $g(x_n)$, such as the one in which the controller is linear in $F(x_n)$ or even x_n (as in traditional feedback), might be more appropriate. Of course when faster convergence is desired, Eq. (7) may be used. Faster convergence comes about as a consequence of the inclusion of $\nabla g(x_n)$ (in the expression for the corrections), which leads to an enlargement of the control region as opposed to straightforward feedback (or the type Pyragas suggested [4]) where the small perturbations control is applicable. Detailed studies of these issues will be given elsewhere.

C. Noisy state variables

An important question concerning any method for control of chaos is its robustness against external noise. This issue is very relevant in practical applications where dealing with noise and its consequences are inevitable. We put our algorithm to the test by applying noise to our numerical simulations (keeping in mind that the optimization is to the lowest order). We found, for reasonable noise levels, control is effective as long as $\Gamma \geqslant c\sigma$, where Γ is the threshold for $g(x_n)$, c a positive constant of order unity, and σ the noise level [9]. As expected, for a small threshold it takes longer for the control to become effective [1], as is shown in Fig. 4 for the logistic map.

V. CONCLUSION

We conclude with a series of remarks intended to illustrate the scope of application and versatility of our approach to the control of chaotic systems.

(i) We construct an error function, the performance index, as a measure of the deviation from the desired behavior. As such, the error function is general in that it needs to include only the performance specifications (e.g., periodicity) and nothing else. This means that if any of the attributes of the desired time evolution can be embodied in an error function, the method we present renders the system controlled by optimizing the constructed error function. A performance index may be any observable, the output intensity of a laser, or the efficiency associated with heat dissipation in a power array (field effect transistors).

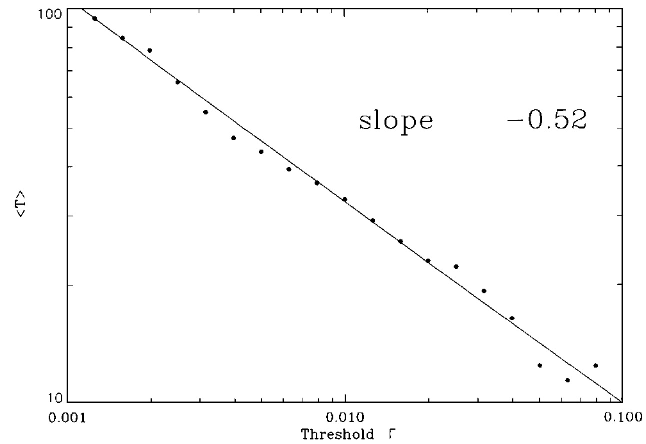


FIG. 4. The average time (the number of iterates in the case of maps) it takes for the control to become effective vs the size of the threshold for $g(x_n), \Gamma$ (for the logistic map).

(ii) Within the framework of our strategy, a similar technique can be applied directly to a system of ordinary differential equations, $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x},p)$. There are in fact two approaches to this problem. First, the Poincaré surface of the section may be used to develop a map, and apply the control to the map directly. Second, by minimizing $g(x,T) = [x(t) - x(t-T)]^2$, the chaotic flow is stabilized to a limit cycle, where T can be found by monitoring the evolution of $g(x,T)$ [12] (detailed discussion will be given in [9]). However, as a special case of stabilizing a steady state, $\mathbf{F} = \mathbf{0}$, we chose the Lorenz system [13]. By minimizing $||\mathbf{F}(\mathbf{x},p)||$, the system was controlled to the steady state at the origin, Fig. 5. This method of control can be used to stabilize orbits not on the attractor, still keeping the control small.

(iii) If an accurate model of the system is not available, we can resort to the methods of embedding [14]. We can reconstruct a local version of the map around points for which $g(x_n)$ is small. The procedure of optimization of $g(x_n)$ can then be applied straightforwardly [15].

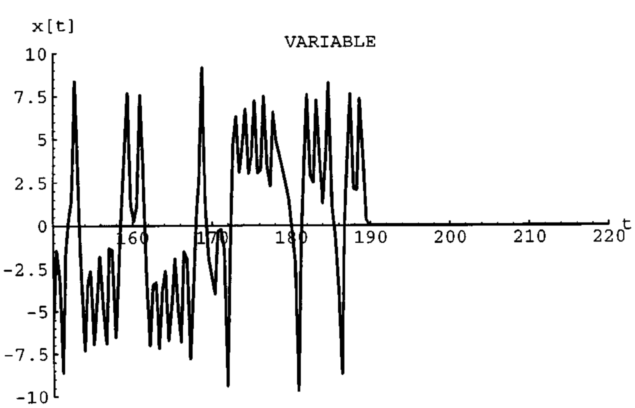


FIG. 5. The Lorenz system is stabilized around the steady state $(x,y,z) = (0,0,0)$ orbit. The control perturbation was smaller than 4% of the extent of the attractor. The control may still be achieved with smaller values of control signal; however, it would take longer.

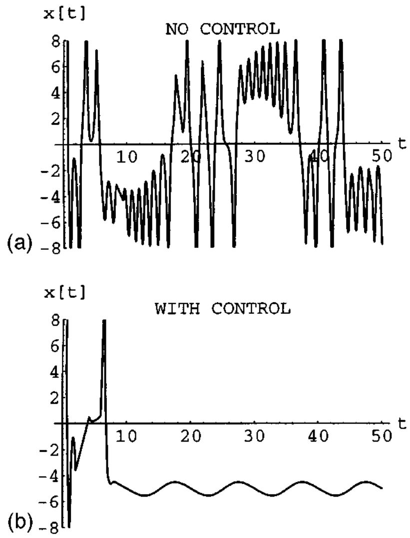


FIG. 6. (a) The unperturbed but driven Lorenz system looks qualitatively similar to the undriven system, however here the periodic orbits are not fixed features of the phase space. (b) In the presence of control, the orbit is very quickly taken to the vicinity of the moving steady state.

(iv) We alluded earlier to the feasibility of applying this method to slowly driven systems (or systems that change ''slowly'' over time). The periodic orbits of a driven system are no longer fixed objects of the phase space, and in fact, for small driving amplitudes for which the structural stability of the system is intact, the periodic orbits move while remaining topologically equivalent to the periodic orbits of the system without driving. We take the Lorenz system [13] and drive one of its parameters sinusoidally in time. The control moves the system ''close'' to the actual driven steady state, Fig. 6. This technique may be relevant in the case of systems similar to the heart [6], for which the system's characteristics change under different operating conditions, hence changing the phase space coordinates of the periodic orbits.

(v) The method of optimal control, as applied to linear systems, is well established in the literature. We believe the methods of optimal control, though not directly applicable to nonlinear systems, may be modified to yield a broader understanding of the mechanisms responsible for the type of periodic behavior brought on by arbitrary perturbations (of the kind introduced in [5]) of a chaotic system. Our preliminary results indicate that using an approach outlined in [7], a strategy very similar to that of standard optimal control may be devised to control a low-dimensional chaotic system to a periodic behavior. A complete reporting and discussion of these results will be given elsewhere [9].

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